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AN EXISTENCE THEOREM FOR THE SOLUTION OF A TYPE OF REAL MIXED DIFFERENCE EQUATION.

BY ALBERT A. BENNETT.

1. **A Mixed Difference Equation.** An equation of the form

$$(1) \quad y_{h+1}^{(0)} = F[x; y_0^{(0)}, y_0^{(1)}, \dots, y_0^{(k)}; y_1^{(0)}, y_1^{(1)}, \dots, y_1^{(k_1)}; \dots; y_h^{(0)}, y_h^{(1)}, \dots, y_h^{(k_h)}],$$

where $y_i^{(j)}$ denotes $(d^j/dx^j)y(x+i)$, is called a mixed difference equation, although not of the most general type. The numbers, $h, k, k_1, k_2, \dots, k_h$, are, of course, non-negative integers, and either k or one of the numbers $k_i, i = 1, 2, \dots, h$ must be positive, in order that the equation may not reduce to an ordinary difference equation. We shall confine ourselves to the particularly interesting case in which the integer k is greater than any one of the integers $k_i, i = 1, 2, \dots, h$, no other restriction being imposed upon these numbers. We shall consider F as any function subject to the following restrictions:

(i) $F[x; u_{00}, u_{01}, \dots, u_{0k}; u_{10}, u_{11}, \dots, u_{1k_1}; \dots; u_{h0}, u_{h1}, \dots, u_{hk_h}]$ is real and finite in the composite real domain of the arguments defined as independent variables for which x is real, finite and non-negative, and each u_{ij} is real and finite but either positive or negative.

(ii) Every partial derivative of F of any order and with respect to any of the arguments, exists in the domain above mentioned.

(iii) $\partial F / \partial u_{0k}$ does not vanish in the domain.

It should be noted that while F is finite, it need not be bounded in the domain.

We shall examine the nature of the solutions of equation (1) for x real and non-negative.

2. **The Continuity of Every Solution and of Its Derivatives.** In the theory of ordinary and of mixed difference equations it is frequently convenient to consider functions $f(x)$ which present singularities for certain isolated values of x . Thus at a point a , $f(x)$ may be undefined, while in a sufficiently small neighborhood on the right $f(x)$ may be defined and continuous, and similarly on the left. If the limit of $f(x)$ exists as x approaches the value a from the right, this limit may be denoted by $f(a+)$. The expression $f(a-)$ may be defined analogously. We may generalize the equation (1) slightly by regarding $y_i^{(j)}$ as denoting $(d^j/dx^j)y(x+i)$

for points at which $y_i^{(j)}$ and its j th derivative exist and are continuous, while permitting also the existence of certain isolated points a for which $y_i^{(j)}$ denotes the limit of $(d^j/dx^j)y(x+i)$ as x approaches a from the right or again as x approaches a from the left. In the usual theory of real difference equations, discontinuous solutions may be constructed with the greatest ease, while solutions continuous at all points on the positive real axis are obtained only when additional boundary conditions are introduced, and fresh boundary conditions must be satisfied for every additional derivative that is required to be continuous. Furthermore by a suitable definition at the points of discontinuity, discontinuous solutions and solutions with discontinuous derivatives of suitable order may be obtained in the usual theory.

That discontinuous functions or functions with no derivatives of order higher than a given number at certain points, may exist which satisfy equation (1) is more or less obvious, provided that the symbols $y_i^{(j)}$ be interpreted in the more general sense above mentioned. We shall however in the following regard y_i^j as originally defined. The existence of a solution is no longer obvious, particularly in view of a theorem that we shall now prove.

For equation (1) to have a meaning, $y(x)$ must exist together with its first k derivatives for any positive values of x , while for the end point $x = 0$, the derivatives are regarded as existing on the right only. In particular for any non-negative point x_0 , $y^{(k)}(x_0)$, $y^{(k)}(x_0 + 1)$, $\dots$, $y^{(k)}(x_0 + h)$, and also $y'(x_0 + h + 1)$ must be supposed to exist. Now (1) when regarded as an implicit equation defining $y^{(k)}(x_0)$, in terms of the other quantities, may be solved for $y^{(k)}(x_0)$, since by hypothesis $\partial F / \partial [y^{(k)}(x_0)]$ does not vanish in the domain. The resulting equation may be differentiated totally with respect to x , and evaluated for $x = x_0$. The form of this last equation in terms of the partial derivatives of F is readily obtained as follows: Let us define G by the identical relation,

$$y_0^{(k)} \equiv G[x; y_0^{(0)}, y_0^{(1)}, \dots, y_0^{(k-1)}; y_1^{(0)}, \dots, y_1^{(k_1)}; \dots; y_h^{(0)}, \dots, y_h^{(k_h)}; F].$$

Differentiating equation (1), we have

$$\begin{aligned} y_{h+1}^{(1)} = & F_x[x; y_0^{(0)}, y_0^{(1)}, \dots, y_0^{(k-1)}, G; y_1^{(0)}, \dots, y_1^{(k_1)}; \dots; y_h^{(0)}, \dots, y_h^{(k_h)}] \\ & + F_{y_0^{(0)}} \cdot y_0^{(1)} + F_{y_0^{(1)}} \cdot y_0^{(2)} + \dots + F_{y_0^{(k-1)}} \cdot G + F_G \cdot G' \\ & + F_{y_1^{(0)}} \cdot y_1^{(1)} + \dots + F_{y_1^{(k_1)}} \cdot y_1^{(k_1+1)} + \dots + F_{y_h^{(0)}} \cdot y_h^{(1)} \\ & + \dots + F_{y_h^{(k_h)}} \cdot y_h^{(k_h+1)}. \end{aligned}$$

where $F_G \equiv F_{y_0^{(k)}} \neq 0$. Solving for G' , we obtain the equation desired in the form

$$y_0^{(k+1)} = G' = \frac{1}{F_{y_0^{(k)}}} [y_{h+1}^{(1)} - F_x + F_{y_0^{(0)}} y_0^{(1)} - \dots - F_{y_0^{(k-1)}} \cdot y_0^{(k)} \\ - \dots - F_{y_h^{(k_h)}} y_h^{(k_h+1)}]$$

where $y_0^{(k)}$ is throughout replaced by its value in terms of the other quantities. The terms of the right hand member are all determinate throughout the entire domain considered, so that $y_0^{(k+1)} = y^{(k+1)}(x_0)$ must be regarded as existent at every non-negative x_0 . It is here that we make use of the fact that k is greater than any one of the integers k_i , $i = 1, 2, \dots, h$. By replacing x_0 by $x_0 + i$ we conclude the existence of $y_i^{(k+1)}$, $i = 1, 2, \dots, h$. The entire reasoning may then be repeated to show the existence of $y_0^{(k+2)}$, etc. By successive applications we prove the existence of each element in the following doubly infinite array as a consequence of the supposed existence of a solution, the array being $y_i^{(k+l)}$, $l = 0, 1, 2, \dots, i = 0, 1, 2, \dots$.

We shall speak of values assumed by the following expressions, as the *initial values* of $y(x)$, viz., $y(x_0)$, $y'(x_0)$, $\dots$, $y^{(k)}(x_0)$; $y(x_0 + 1)$, $y'(x_0 + 1)$, $\dots$, $y^{(k_1)}(x_0 + 1)$; $\dots$; $y(x_0 + h)$, $y'(x_0 + h)$, $\dots$, $y^{(k_h)}(x_0 + h)$, where x_0 takes on all values for which $0 \leq x_0 < 1$. The initial values are therefore infinite rather than finite in number. We now have the theorem:

THEOREM. *Every solution of (1), having real and finite initial values, is continuous with all of its derivatives at all finite points on the positive real x -axis.*

3. Power Series Representing a Solution. It will be granted that there exist equations of the form (1) having solutions. We shall for the moment consider the nature of a power series representing a solution supposed to exist without entering into the question as to when such a solution exists. When once the form of the solution is known, we will be prepared to prove that a solution always exists.

Let us consider a given function $y(x)$, which is continuous with all of its derivatives on the positive real axis. At any point of the positive real axis, the function $y(x)$ may be expanded as a Taylor's series in finite form, where the degree of the highest term in $(x - x_1)$ is arbitrarily preassigned. We are led to an infinite Taylor's series at the point x_1 , which will however in general be divergent for every value of x different from x_1 . The function completely determines the series, and the series completely determines the value of the function and of all of its derivatives at the point x_1 . We may therefore regard the series, even when divergent as representing the function at the point x_1 , although perhaps not representing the function at any other point. The series while determining the differential coef-

ficients may not determine a function element. If $S(x)$ denote the series, we may write

$$y(x) \sim S(x) \equiv a_0 + a_1(x - x_1) + \cdots + a_n(x - x_1)^n + \cdots.$$

As is usual* in the theory of divergent power series if $y_1(x) \sim S_1(x)$, and $y_2(x) \sim S_2(x)$, then $(y_1 + y_2) \sim (S_1 + S_2)$, $(y_1 \times y_2) \sim (S_1 \times S_2)$ and if (y_1/y_2) is finite $(y_1/y_2) \sim (S_1/S_2)$ also $y_1^n \sim S_1^n$, where the operations are performed formally, as with convergent power series. In the usual theory, the series S is regarded as initially given, and the function y as determined from S . It then occurs that in general we cannot say that $y' \sim S'$ where y' is the derivative of $y(x)$ at $x = x_1$, and S' is obtained by differentiating S term by term. When we use the fact that, as in the present case, $y(x)$ is given, and is continuous together with all of its derivatives in the neighborhood of $x = x_1$, we see from the nature of Taylor's series, that $y^{(m)} \sim S^{(m)}$, for every positive integer m . Hence we conclude that if equation (1) has a solution $y(x)$ with real finite initial values, $y(x)$, can be represented at every point $x_1 \geq 0$, by a series S , in powers of $x - x_1$, which is such that $y \sim S(x)$, and $y^{(m)} \sim S^{(m)}(x)$, where m is any positive integer.

The power series which represents a solution y , at one of the points $x_1 = x_0, x_0 + 1, \cdots, x_0 + h$, will be spoken of as an *initial series* of y , for x_0 . For the given point x_1 , and the given solution y , the initial series is, of course, unique. For a given x_0 , there will be for y , $h + 1$ initial series corresponding to $x_1 = x_0, x_0 + 1, \cdots, x_0 + h$. From the nature of the equation (1), the series representing a solution $y(x)$, at a point $x_1 = x_0 + d$ where d is a positive integer greater than h , will be determined completely by the $h + 1$ initial series of y for x_0 , and the only restrictions upon the behavior of y between x_0 and $x_0 + 1$, are the end conditions requiring y and its derivatives, to approach the given values at x_0 , and x_1 , and the general conditions that the function and its derivatives be continuous.

4. The Existence of Solutions with Assigned Initial Series. The situation arising with regard to the number of arbitrary conditions that may be imposed upon a solution of the equation (1) is in marked distinction to that occurring, for example, in the usual theory of linear differential equations. In the first place the initial series for x_1 and x_2 where $0 \leq x_1 < x_2 < 1$, are entirely independent of each other, a fact which we have already mentioned. This is a feature common to all difference equations. The noteworthy fact concerning the equation (1) may be stated as follows:

* Cf. Borel, *Leçons sur les séries divergentes*, Borel Series, and Van Vleck, *Selected Topics*, etc., Boston Colloquium.

THEOREM. *Any $h + 1$ series, divergent if desired, may be chosen arbitrarily as the initial series for $x = 0$, of a solution of equation (1). There will then exist an infinite number of distinct functions, $y(x)$, all of which satisfy (1), which have the given series as initial series and which are continuous with all of their derivatives for all positive real values of x . For each positive integral value d of x , there will be a series uniquely determined by the initial series and the point d which is the common Taylor's series for all of these solutions, $y(x)$.*

The above theorem will be demonstrated, if we can construct a function $f(x)$ defined for x in the interval $0 \leq x < h + 1$, with the assigned initial series at the points $x = i$, $i = 0, 1, \dots, h$, and with a series representation at $x = h + 1$, identical with that determined by (1) from the assigned initial series, the function $f(x)$ being continuous together with all of its derivatives within and at the extremities of the interval. For the equation (1) will then determine the value of $y(x)$ and of its derivatives at $x = h + 1$, and these will coincide with the values already determined for this point, if $y(x)$ be defined as equal to $f(x)$ for $0 \leq x < h$. The equation (1) will then define $y(x)$ over the entire positive real axis. The function $f(x)$ is further supposed to contain at least one arbitrary constant. We shall prove the existence of such a function by exhibiting a function $f(x)$ which satisfies not only these but also more stringent conditions.

5. The Existence of a Function with Assigned Series at an Infinite Number of Points. We shall now prove the following theorem:

THEOREM. *If any point x_0 on the real x -axis be selected, there exists a real function $f(x)$, containing an infinite number of arbitrary real parameters, but which regarded as a function of x is continuous together with all of its derivatives at every finite real point of the x -axis for every choice of the parameters. Also the Taylor's series of $f(x)$ at each of the infinite array of points, $x = x_0 + i$, $i = \dots, -2, -1, 0, 1, 2, \dots$, is independent of the values of the parameters, being previously assigned for each of these points independently, without regard to convergence or the behavior at other points.*

To demonstrate the truth of this theorem, we shall first exhibit a function $f_i(x)$ which is finite and continuous for all real finite values of x , which vanishes with all of its derivatives at the points $x = x_0 + i + j$, $j = \dots, -2, -1, 1, 2, 3, \dots$, and which has at the point $x = x_0 + i$, the preassigned Taylor's series associated with this point, viz.,

$$a_{i,0} + a_{i,1}(x - x_0 - i) + a_{i,2}(x - x_0 - i)^2 + \dots \\ + a_{i,n}(x - x_0 - i)^n + \dots$$

The function $f_i(x)$ will itself be obtained in the form $f_i(x) \equiv \sum_{n=0}^{\infty} u_{i,n}$,

where

$$u_{i,n} \sim 0 + \cdots + 0 + a_{i,n}(x - x_0 - i)^n + 0 + \cdots.$$

The final function $f(x)$ will be defined as $\sum_{i=-\infty}^{\infty} f_i(x)$. Each of the functions $u_{i,n}$ will involve in its definition an arbitrary real positive parameter $b_{i,n}$.

6. The Definition of $u_{i,n}$. The function $u_{i,n}$ may be defined in numerous different manners. Perhaps the simplest definition is suggested by the function constructed by Mr. J. F. Ritt.* He has noted that the function

$$v_n = \frac{a_n z^n}{n!} (1 - e^{-(1/b_n z^2)})^{p_n},$$

where $p_n < b_n > 2^{p_n-1}|a_n|$, and where p_n is always an integer not less than one, is such that $\sum_{n=1}^{\infty} v_n$ converges uniformly, and absolutely together with all of its derivatives at every point $z = x + iy$ for which $y^2 \leq m^2 x^2$, where $m^2 < 1$. On the other hand the function represented in the region considered will have at the origin the number a_n as the n th differential coefficient, for methods of approach which lie in the region. The series of numbers a_n may be assigned as arbitrary complex quantities without regard to convergence.

When we form the product

$$\frac{1}{1 + e^{\frac{4z^2-1}{z^2(1-z^2)}}} \sum v_n = g(z),$$

we obtain a function having the same differential coefficients at the origin when approached in the above region, and whose corresponding coefficients at $z = 1$, when approached from the left and $z = -1$, when approached from the right are all zero. For, indeed, the factor $1/(1 + e^{(4z^2-1)/z^2(1-z^2)})$ is finite in the region considered, while its expansions at the origin, the point $z = 1$, and the point $z = -1$ are respectively those of $(1 - e^{-(1/2^2)})$, $(e^{-(1/1-2)})$, and $(e^{-(1/1+z)})$ owing to the peculiarity of the function $e^{-(1/z^2)}$ at the origin. We are concerned only with the real axis in the present discussion. We may now define $u_{i,n}$ as follows:

$$u_{i,n} = \frac{a_{i,n} \{1 - e^{-\frac{1}{b_{i,n}[x-(x_0+i)]^2}}\}^{p_{i,n}}}{1 + e^{\frac{4[x-(x_0+i)]^2-1}{[x-(x_0+i)]^2\{1-[x-(x_0+i)]^2\}}}} x^n, \quad \text{for} \quad |x - (x_0 + i)| \leq 1,$$

* *Annals of Mathematics*, vol. 18, pp. 18-23.

$u_{i, n} \equiv 0$ for $|x - (x_0 + i)| \geq 1$, $-\infty < i < \infty$, $0 \leq n < \infty$, where $b_{i, n}$ is any real positive constant such that, $b_{i, n} > 2^{p_{i, n-1}} |n! a_{i, n}|$, and $p_{i, n}$ is any integer, > 0 , $< b_{i, n}$.

With this definition, $f_i(x) = \sum_{n=0}^{\infty} u_{i, n}$ is a real function of x , continuous with all of its derivatives along the entire real axis and vanishing with all of its derivatives for $|x - (x_0 + i)| \geq 1$, while the Taylor's series for $f_i(x)$ at $x = x_0 + i$ is simply $\sum_{n=0}^{\infty} a_{i, n} [x - (x_0 + i)]^n$.

The series $\sum_{i=-\infty}^{+\infty} f_i(x)$ will then be absolutely and uniformly convergent for all real values of x , with an ascribed Taylor's series at each point $x_0 + i$. $-\infty < i < +\infty$, i , an integer. Also the function $f(x)$, represented by $\sum_{i=-\infty}^{+\infty} f_i(x)$, is continuous together with all of its derivatives at every real finite point, which establishes the theorem.

7. Some References to the Recent Literature. The following references to the most recent literature concerning questions related to those here discussed may prove of interest.

Mixed Difference Equations.

O. Polusuchin, Diss., Zürich, 1900.

E. Schmidt, Math. Ann., 70 (1911), p. 499 ff.

F. Schürer, Ver. d. K. Sächs, Ges. d. Wiss., 64 (1912), p. 167-236, and 66 (1914), p. 137-159.

H. Bateman, Proc. Fifth Int. Cong. Math. (1912), p. 291-294.

Functions with Assigned Taylor's Series

E. Borel, Leçons sur les Fonctions de Var. Réelles, Borel Ser., 1905.

S. Bernstein, Note in d'Adhemard's Leçons de l'Analyse, II, p. 239-283.

J. F. Ritt, *Annals of Math.*, loc. cit.

PRINCETON UNIVERSITY,
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